Introduction

In this supplement, we describe how to construct positional and translational functions for satisfying the seam constraints. We assume that we are given a mesh $M$ and a set of (oriented) seams $\sigma \in \Sigma$ with associated rotations $r_\sigma \in \{1, i, -1, -i\}$ satisfying $r_{\sigma^{-1}} = r_{\sigma}^{-1}$. For a given choice of consistent translations shifts across seams, $\{t_\sigma\} \subset \mathbb{C}$ (with $t_\sigma^{-1} = -r_{\sigma}^{-1}t_\sigma$), our goal is to define a function space making it easy to express functions $F : M \to \mathbb{C}$ that satisfy the following condition:

**Seam Constraint (Mesh):** For any point $p$ on any seam $\sigma$, the limits of the function values as we approach the point $p$ from below and above the seam satisfy:

$$
\lim_{p^- \to p} F(p^+) = r_\sigma \left( \lim_{p^- \to p} F(p^-) \right) + t_\sigma,
$$

where $t_\sigma$ is the prescribed translation across $\sigma$.

Local Domain Graph

We consider a single basis function $B : M \to \mathbb{C}$, suppressing its index for simplicity as in the paper. This function is supported on the open, connected region $D$, which we cut into components with the seams in $\Sigma$:

$$
\bigcup \limits_i D_i = D - \bigcup \limits_{\sigma \in \Sigma} \sigma.
$$

Using region $D_i$’s indicator function $\chi_i(p)$, we can decompose $B(p)$ into a sum of individual basis functions $B_i(p) = \chi_i(p)B(p)$ as in the paper.

We define $D$’s local domain graph, $\mathcal{G}_D$, with nodes $v \in V(\mathcal{G}_D)$ corresponding to the components $D_i$, and with edges $\{v, w\}$ for any two regions sharing a seam (i.e., $D_v \cap D_w \neq \emptyset$). $\mathcal{G}_D$ is a directed graph since we care about the direction in which we cross a seam, but because its edges always come in pairs $e = \{v, w\}$, $e^{-1} = \{w, v\}$, we pretend that it is undirected when discussing spanning trees and cycles.

For convenience, we consider functions on $V(\mathcal{G}_D)$’s vertices that are equivalent to the mesh functions spanned by $B_i$ due to the mapping:

$$
\begin{align*}
\{V(\mathcal{G}_D) \to \mathbb{C}\} & \to \{M \to \mathbb{C}\} \nonumber \\
F(v) & \mapsto F(p) = \sum \limits_{v \in V(\mathcal{G}_D)} F(v)B_i(p). \tag{1}
\end{align*}
$$

Now we can replace the seam constraints for a function on the mesh with the constraint that $F : V(\mathcal{G}_D) \to \mathbb{C}$ satisfies:

**Seam Constraint (Graph):** For any edge $\{v, w\} = e \in E(\mathcal{G}_D)$ we must have:

$$
F(w) = r_{\ell(e)}F(v) + t_{\ell(e)},
$$

where $\ell(e) \in \Sigma$ is the seam corresponding to edge $e$.

### Notation:

Given a path $\pi \subset E(\mathcal{G}_D)$ and given an edge along the path, $e \in \pi$, we denote by $\pi^e$ and $\pi^{-e}$ the two halves of the path on either side of $e$:

$$
\pi = \pi^e \circ e \circ \pi^{-e}.
$$

### Definition:

We define the product of rotations along the path $\pi \subset E(\mathcal{G}_D)$ as:

$$
R(\pi) = \prod \limits_{e \in \pi} r_{\ell(e)}.
$$

For consistency, we set $R(\emptyset) = 1$.

**Definition:** Similarly, for a given seam $\sigma \in \Sigma$, we define:

$$
T^\sigma(\pi) = \sum \limits_{e \in \pi} R(\pi^e)\delta_{\sigma,e} \text{ with } \delta_{\sigma,e} = \begin{cases} 1 & \text{if } \sigma = \ell(e) \\ -r_{\sigma}^{-1} & \text{if } \sigma = \ell(e^{-1}) \\ 0 & \text{otherwise} \end{cases}.
$$

For consistency, we set $T^\sigma(\emptyset) = 0$.

#### Proof of Proposition 1:

Letting $\pi \circ \pi_1$ denote the concatenation of two paths, and letting $\pi^{-1}$ denote the reverse of a path, we have:

$$
\begin{align*}
R(\pi_2 \circ \pi_1) = R(\pi_1) \cdot R(\pi_2) \\
T^\sigma(\pi_2 \circ \pi_1) = R(\pi_1)T^\sigma(\pi_2) + T^\sigma(\pi_1)
\end{align*}
$$

Thus, the definitions of $R(\pi)$ and $T^\sigma(\pi)$ are homotopy-invariant.

#### Proof of Proposition 2

Once we know graph function $F$’s value on some vertex $u \in \mathcal{G}_D$, the seam constraints for the edges of a spanning tree, $\mathcal{G}_D$, determine the values at all $v \in \mathcal{G}_D$ because $\mathcal{G}_D$ is strongly connected. We show that these values are given by

$$
\tilde{F}(v) = c_v\tilde{F}_u(v) + \sum \limits_{\sigma \in \Sigma} t_\sigma \tilde{F}_u^\sigma(v), \tag{3}
$$

with positional and translational functions $\tilde{F}_u$ and $\tilde{F}_u^\sigma$ defined below. Of course, when $\mathcal{G}_D \neq \mathcal{G}_D$ (because of cycles), there will be an additional constraint added by each “undirected” edge not in this spanning tree, but this is not considered for Proposition 2.

Traversing a path $\pi_{uv}$ in $\mathcal{G}_D$ from $u$ to $v$, we accumulate a rotation of $R(\pi_{uv})$ and a contribution of $T^\sigma(\pi_{uv})$ to $\sigma$’s coefficient. This motivates the following definitions:

$$
\tilde{F}_u(v) = R(\pi_{uv}), \quad \tilde{F}_u^\sigma(v) = \frac{T^\sigma(\pi_{uv})}{2}, \tag{4}
$$

which indeed satisfy the constraints as we prove in Lemma 1. Since all paths on tree $\mathcal{G}_D$ are homotopy equivalent and $R(\pi_{uv})$ and $T^\sigma(\pi_{uv})$ are homotopy-invariant, these functions are uniquely defined. The division by two is needed because of our redundant use of both $t_\sigma$ and $t_\sigma^{-1}$: for every edge $e \in \pi_{uv}$, we accumulate not only translation $t_{\ell(e)}$ but also the identical translation $-r_{\ell(e)}t_{\ell(e^{-1})}$.

**Lemma 1:** Function $\tilde{F}(v)$ in (3) with positional and translational functions from (4) satisfies every seam constraint in $\mathcal{G}_D$.  

1Note that this formulation implicitly assumes that the basis functions form a partition of unity.
Proof. Taking any \( \{v, w\} = e \in E(\tilde{S}_D) \),
\[
F(w) = c_u R(e \circ \pi_{wu}) + \sum_{\sigma \in \Sigma} t_\sigma R(\pi_{wu}) + \frac{T^\sigma(\pi_{wu})}{2} c_u R(e \circ \pi_{wu}) + \frac{T^\sigma(\pi_{wu})}{2} \cdot \frac{t_\sigma - r_\sigma t_\sigma}{2}.
\]
\[
= r_\sigma(c_u R(e \circ \pi_{wu}) + \sum_{\sigma \in \Sigma} t_\sigma R(\pi_{wu}) + \frac{T^\sigma(\pi_{wu})}{2} + t_\sigma - r_\sigma t_\sigma).
\]
so we see that the edge’s constraint is satisfied. \( \Box \)

Completing the Proof: Picking \( u = 0 \), corresponding to the reference domain \( D_0 \), and applying mapping (1), the constrained mesh function can be written as:
\[
F(p) = \sum_{j} c_0 R(\pi_{j0}) + \sum_{\sigma \in \Sigma} t_\sigma R(\pi_{j0}) B_j(p),
\]
\[
= c_0 \sum_{j} R(\pi_{j0}) B_j(p) + \sum_{\sigma \in \Sigma} \sum_{j} t_\sigma \left[ \frac{T^\sigma(\pi_{j0})}{2} B_j(p) \right],
\]
revealing \( \tilde{B} \) and \( \tilde{B}_{loc} \) as presented in Proposition 2.

Accounting for Cycles in \( \tilde{S}_D \)

The constraints on edges \( E^{gy} := E(\tilde{S}_D) \setminus E(S_D) \) (if any exist) generally remain unsatisfied by (3): for \( \{v, w\} = e \in E^{gy} \),
\[
r_\sigma(c_u R(e \circ \pi_{wu}) + \sum_{\sigma \in \Sigma} t_\sigma R(\pi_{wu}) + \frac{T^\sigma(\pi_{wu})}{2} + t_\sigma - r_\sigma t_\sigma).
\]
However, we can construct a new graph function \( F \) satisfying this constraint. The clearest way is to first traverse back to the source vertex \( u \) on both sides of the constraint by following \( \pi_{wu}^{-1} \):
\[
c_u R(\pi_{wu}^{-1} \circ e \circ \pi_{wu}) + \sum_{\sigma \in \Sigma} t_\sigma R(\pi_{wu}^{-1} \circ e \circ \pi_{wu}) + \frac{T^\sigma(\pi_{wu}^{-1} \circ e \circ \pi_{wu})}{2} + t_\sigma - r_\sigma t_\sigma = F(w).
\]
Rearranging, we arrive at the linear equality constraint on the positional and translational coefficients:
\[
c_u(R(\zeta_e) - 1) + \sum_{\sigma \in \Sigma} t_\sigma \frac{T^\sigma(\zeta_e)}{2} \zeta_e = 0,
\]
where \( \zeta_e = \pi_{wu}^{-1} \circ e \circ \pi_{wu} \) is the cycle formed by adding edge \( e \) to \( \tilde{S}_D \). Note that if \( R(\zeta_e) = 1 \), positional coefficient \( c_u \) disappears from the constraint, and we’re left with a constraint on the translation variables, \( t_\sigma \), only. However, when \( R(\zeta_e) \neq 1 \), we choose to eliminate \( c_u \) using the constraint:
\[
c_u = \frac{1}{1 - R(\zeta_e)} \sum_{\sigma \in \Sigma} t_\sigma \frac{T^\sigma(\zeta_e)}{2} T^\sigma(\zeta_e).
\]
In doing this, we remove \( \tilde{B} \) (mesh function \( \tilde{B} \)) as a distinct basis function, but we fold its contribution into the translational basis functions that \( c_u \) depends on.

We repeat this procedure for every \( e \in E^{gy} \), collecting all translational coefficient constraints into a linear system. We then solve for a set of independent translations, \( t_n \) for \( n \in I_{ind} \), determining dependent translations \( t_d \) for \( d \in I_{dep} \) by a linear combination:
\[
t_d = \sum_{n \in I_{ind}} w_{mn} t_n.
\]
For convenience, we extend this weight matrix \( W \) to compute all translations \( t_m \) for \( m \in I_{ind} \setminus I_{dep} \) by adding rows \( w_{mn} = \delta_{mn} \) (Kronecker delta) for \( m, n \in I_{ind} \).
If all cycles have \( R(\zeta_e) = 1 \), our final graph function \( F \) is given by expressing \( F \) in terms of the independent translations \( t_n \) for \( n \in I_{ind} \):
\[
F(v) = c_u \tilde{F}_u(v) + \sum_{m \in I_{ind}} \left[ \sum_{n \in I_{ind}} w_{mn} t_n \right] \tilde{B}^m(v).
\]
Here we used the fact that each translational coefficient \( t_m \) corresponds to some seam \( \sigma_m \). If any of the cycles has \( R(\zeta_e) \neq 1 \), eliminating the positional coefficient \( c_u \) gives:
\[
F(v) = \sum_{m \in I_{ind}} \left[ \sum_{n \in I_{ind}} t_n w_{mn} \right] \tilde{B}^m(v) + \frac{1}{1 - R(\zeta_e)} T^\sigma(\zeta_e) \tilde{F}_u(v).
\]
By mapping these into mesh functions and repeating for each general basis function \( B \), we arrive at Proposition 3.